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A Stochastic Model of a Quantum Field Theory

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The problem of obtaining a realistic relativistic description of a quantum system is discussed in the context of a simple (light-cone) lattice field theory. A natural stochastic model is proposed which, although nonlocal, is relativistic (in the appropriate lattice sense), and which is operationally indistinguishable from the standard quantum theory. The generalization to a broad class of lattice theories is briefly described.

KEY WORDS: Lattice quantum field theory; realistic model; relativistic formulation; stochastic process.

1. INTRODUCTION

Quantum theory is a highly successful algorithm for predicting the results of experiments. It is, however, beset with worrying conceptual problems. These may be traced to the fact that although one may calculate probabilities, there is no objectively defined space of events to which these probabilities refer. In a typical interpretation, a space of events must first be chosen from an infinite number of incompatible alternatives, and probabilities may only be extracted after this choice is made. The choice —whether it is effected by means of a division of the world into system and observer, or system and environment; or whether it is a choice of a subalgebra of observables, or one among many sets of consistent histories—is still in the end a subjective choice. Its role seems inappropriate in what is supposed to be a fundamental theory, and to beg the question of how to explain the very particular space of events that constitutes the world of our actual experience, namely the classical one.(1)

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To a realist, the form that a remedy must take is clear: a space of objectively defined events must be restored to the theory. If difficulties are encountered in pursuing this course, then they will be entirely conventional ones—those of meeting the demands of predictive power, simplicity, consistency, and on—but understanding the relationship of the theory to the experience it is meant to describe should not pose special philosophical problems.

Realistic theories have essentially been of two types: either the wavefunction evolves according to the standard unitary law and is supplemented by some further variables with prescribed dynamics, as, for instance, in the theory of de Broglie and Bohm[2] and Bell's stochastic generalization to quantum field theory[3,4]; or alternatively, a definite departure from quantum theory is contemplated and the wavefunction is subject to a stochastic and nonunitary evolution (see, e.g., Pearle,[5] Percival,[6] and references therein). Of particular practical interest is that theories of the second type, by predicting deviations from standard results, are beginning to suggest experimental tests of quantum theory itself.

As is well known from the work of Bell,[1,7] realistic theories must inevitably be nonlocal in character. The purpose here is to examine the question of whether, despite this nonlocality, it is possible to preserve relativistic invariance. More precisely, we would like a theory that is "fundamentally" relativistic: not only should it enjoy phenomenological Lorentz invariance (as, for instance, the theory of Lorentz himself), but in addition, its formulation should not rest on the choice of a preferred frame.

The proper setting for this question is quantum field theory and the discussion will be based on a particularly simple example—a light-cone lattice field theory in one space dimension—though, as we remark at the end, the generalization to a large class of theories is straightforward. On the lattice of course one cannot have full Lorentz symmetry. However, there is a causal structure and the notion of a spacelike surface, and in this restricted context a "relativistic theory" shall mean one in which the dynamics does not depend on a preferred choice of such surfaces. For an appropriate lattice theory one expects to recover full Lorentz invariance in the continuum limit.

We shall show that the most naive attempt at a realistic formulation fails, and then propose a rather natural model with the desired properties. The model is of the first type mentioned above: thus the mathematical structure of quantum theory is left completely intact, but is supplemented by extra variables governed by a stochastic evolution law. It is very much an attempt at a minimal solution to the problem and agreement with the results of conventional quantum theory is built in, although one might also regard it as the starting point for a theory that differs from the conventional theory in a testable way. Perhaps of particular interest is the generality of the construction: essentially the only features of the underlying quantum theory that it requires are the causal structure and the local, unitary evolution law. One may thus obtain a straightforward probabilistic description of a rather general class of local lattice quantum theories.

For some related ideas, and some similar points regarding measurement, see the discussion of how the Bohm theory might be made relativistic by Dürr *et al.*[8] and Berndl *et al.*[9] (and also refs. 10 and 11). For another approach, in the context of a theory of the second type, see Ghirardi *et al.*[12]

2. THE QUANTUM THEORY

Light-cone lattice theory has been introduced in the study of integrable models in $(1+1)$ dimensions;[13] in statistical mechanics the analog is the diagonal-to-diagonal transfer matrix method. It exhibits in a particularly transparent form the essential elements of any local quantum field theory—a causal structure, and a local unitary evolution law adapted to that structure.

The spacetime is represented by a lattice generated by null rays as shown in Fig. 1, and the local observables of the theory live on the links. In the simplest theory there are just two states associated with each link l, labeled by $\alpha_l = 0, 1$, which we shall refer to as the occupation number. At each vertex of the lattice the local evolution law is encoded in a four-dimensional unitary matrix—an "R-matrix"—whose entries, $R_{\alpha_1 \alpha_2}^{\alpha_{1'} \alpha_{2'}}$, are the amplitudes connecting the four possible states on the ingoing and outgoing pairs of links at that point (see Fig. 2a). The causal structure is the obvious one: two links are spacelike separated (the corresponding local operators on the links commute) if and only if there is no everywhere future-directed path on the lattice connecting them.

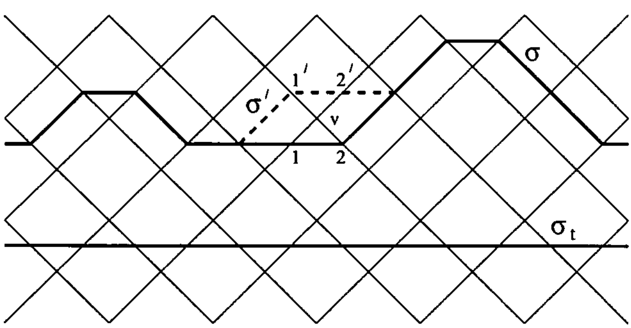

Fig. 1. The light-cone lattice. σ_t is a constant-time slice; σ is a general spacelike surface, and σ' one obtained from it by an elementary motion.

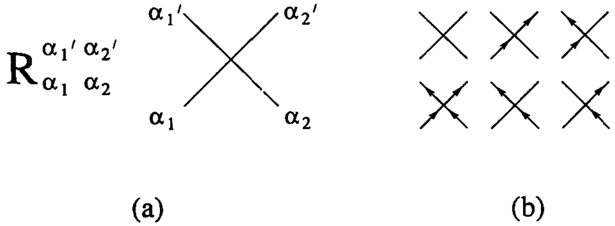

Fig. 2. (a) The R-matrix associated with a vertex. (b) A possible set of nonzero amplitudes conserving occupation number. An arrow on a link denotes occupation number one.

A quantum state Ψ is fully determined by a complex function (the wavefunction) of the variables on the links cut by a constant-time slice σ_t. Denoting this set of variables by $\alpha|_{\sigma_t}$, we write the wavefunction as $\Psi(\alpha|_{\sigma_t})$. The unitary evolution to the wavefunction on the next constant-time slice is then effected by multiplying by all R-matrices associated with the vertices lying just to the future of σ_t, and summing over the repeated α_l's.

More generally, one may consider the wavefunction $\Psi(\alpha|_\sigma)$ on any spacelike surface σ. To evolve it to another surface σ', one now applies all the R-matrices associated with the vertices in the region between σ and σ'. (We assume that σ' is everywhere coincident with, or to the future of, σ, although transformations between intersecting surfaces can be considered just as well.) In the simplest case, when the region contains a single vertex, v say, we shall call the local deformation of σ to σ' an "elementary motion," and only one R-matrix need be applied. Thus, if $l = 1, 2$ and $1', 2'$ are the ingoing and outgoing links, respectively, at v, as shown in Fig. 1, the evolution of the wavefunction is given by

$$\Psi(\alpha|_{\sigma'}) = \sum_{\alpha_1 \alpha_2} R^{\alpha_{1'}\alpha_{2'}}_{\alpha_1 \alpha_2} \Psi(\alpha|_\sigma) \tag{1}$$

The evolution between two arbitrary surfaces can be obtained by composing elementary transformations of this type. To fix the boundary conditions, we take the lattice to be periodic in space, of width $2N$; the spacetime is then a (discretized) cylinder and the wavefunction on any surface is a function of $2N$ variables.

The standard interpretation of the theory is expressed in terms of the results of measurements of arbitrary Hermitian operators associated with any surface σ. It is sufficient, however, to restrict to the projection operators corresponding to the joint occupation number eigenstates labeled by $\alpha|_\sigma$. The point of view represented by this restriction is essentially the familiar one that, ultimately, all measurements reduce to measurements of position. The predictions of the theory are then summarized by the rule that, in a state Ψ, the probability $p_\Psi(\alpha|_\sigma)$ of finding the configuration $\alpha|_\sigma$ on the surface σ is given by

$$p_\Psi(\alpha|_\sigma) = |\Psi(\alpha|_\sigma)|^2 \tag{2}$$

In what follows we shall use the phrase "standard quantum theory" to mean the mathematical formalism together with this standard interpretation, though of course it should be borne in mind that the latter, relying as it does on the notion of measurement, is not very precise. Note that for an incomplete set of spacelike-separated variables $\alpha_{l_1},..., \alpha_{l_n}$ $(n < 2N)$, the joint probability may be calculated as the marginal distribution of (2) for any σ that cuts the corresponding links $l_1,..., l_n$; that this distribution is independent of the choice of σ, and so is well defined, is an immediate consequence of the local unitary evolution of the wavefunction.

This completes then our description of the lattice quantum field theory. Note that we have not yet made any particular assumption about the R-matrices, though for a conventional field theory with spacetime translation invariance they will be uniform over the lattice. With an appropriate choice, and taking a suitable continuum limit, one obtains, for example, the massive Thirring model. The nonzero amplitudes for this case are depicted in Fig. 2b, where occupation number one is indicated by a forward-pointing arrow. One may think of such arrows as the paths of "bare fermions" through the lattice, though these are not to be identified with the physical particles of the eventual continuum theory, since they are built on the wrong vacuum. For further details the reader is encouraged to consult Destri and de Vega.[13] Here we need only note that we have a lattice system that is rich enough to yield a nontrivial quantum field theory in the continuum limit.

3. REALISTIC FRAMEWORK

Let us now try to construct a realistic model of the system. From the point of view of the quantum theory there is nothing particularly special about the variables α_l; transformation theory allows the choice of many other sets of (generally nonlocal) variables just as well. However, for a realistic theory it is natural to take the α_l as fundamental, and to elevate them, in Bell's terminology, to the status of "beables"—in other words, to suppose that in the time evolution of the system each variable α_l realizes a definite value $\hat{\alpha}_l$ with some probability, as part of an objectively defined physical process. More precisely, given a state Ψ, we suppose that there is an associated joint probability distribution $p_\Psi(\hat{\alpha})$ for the realized values $\hat{\alpha}_l$ on the entire spacetime lattice. [To avoid difficulties of definition of $p_\Psi(\hat{\alpha})$, we may restrict attention to a finite number of variables. Thus in what follows $p_\Psi(\hat{\alpha})$ should be regarded as the distribution associated with the variables between two bounding surfaces, with the understanding that these may be moved arbitrarily far into the past and future, respectively.]

To secure agreement with standard quantum theory in this framework, and avoid any reference to a particular frame, it seems simplest to require that all the quantum mechanical probabilities (2) arise as the appropriate marginal distributions of $p_\Psi(\hat{\alpha})$, so that for *all* surfaces σ,

$$\sum_{\hat{\alpha}|_{\sigma^c}} p_\Psi(\hat{\alpha}) = |\Psi(\hat{\alpha}|_\sigma)|^2 \tag{3}$$

where σ^c denotes all those links of the lattice not cut by σ. However, this seemingly natural procedure is too naive. Indeed, it is one of the remarkable properties of quantum theory that in general no such $p_\Psi(\hat{\alpha})$ can be found.

This result follows immediately from the interpretation of the Bell inequalities as conditions for the existence of a joint distribution.[2] It suffices to consider the simple arrangement shown in Fig. 3, with variables α_1, $\alpha_{1'}$, α_2, $\alpha_{2'}$ at just four sites—by an appropriate choice of R-matrices this may be easily embedded in the lattice field theory. The site $1'$ lies in the causal future of 1, the corresponding amplitudes are summarized in the unitary matrix $R(1)_{\alpha_1}^{\alpha_{1'}}$, and in a spacelike-separated region we have a similar arrangement for the 2-variables. There are four spacelike surfaces that can be drawn through the sites and thus four probability distributions $p(\hat{\alpha}_1, \hat{\alpha}_2)$, $p(\hat{\alpha}_1, \hat{\alpha}_{2'})$, $p(\hat{\alpha}_{1'}, \hat{\alpha}_2)$, and $p(\hat{\alpha}_{1'}, \hat{\alpha}_{2'})$ for which the standard quantum rule (2) provides predictions. For these distributions to be obtainable as the marginals of an overall distribution $p(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_{1'}, \hat{\alpha}_{2'})$, it is necessary (and in fact also sufficient) that they satisfy the inequalities[14]

$$-1 \leqslant p(\hat{\alpha}_1, \hat{\alpha}_2) - p(\hat{\alpha}_1, 1-\hat{\alpha}_{2'}) - p(\hat{\alpha}_{1'}, \hat{\alpha}_2) - p(1-\hat{\alpha}_{1'}, \hat{\alpha}_{2'}) \leqslant 0 \tag{4}$$

However, for an appropriate choice of state Ψ and matrices $R(i)$, the predictions of (2) violate these inequalities. For instance, take the state $\Psi(\alpha_1, \alpha_2) = (1/\sqrt{2})(\delta_{\alpha_1 1}\delta_{\alpha_2 0} - \delta_{\alpha_1 0}\delta_{\alpha_2 1})$, and let $R(1) = \exp \frac{1}{2} i\theta\sigma_1$ and $R(2) = \exp -\frac{1}{2} i\theta\sigma_1$, where σ_1 is the first Pauli matrix. Then with $\hat{\alpha}_1 = 0$, $\hat{\alpha}_2 = 0$, $\hat{\alpha}_{1'} = 1$, $\hat{\alpha}_{2'} = 0$, the above combination of probabilities is, according to (2), $-\frac{1}{2}(2 + \cos\theta - \cos^2\theta)$, which violates the inequality for $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$. This is, of course, the familiar mathematics of the Einstein–Podolsky–Rosen–Bohm experiment, albeit with a somewhat different interpretation.

[2] The equivalent result in the context of a realistic particle mechanics has also been shown by Berndl *et al.*[9,10] (making use of an inequality of Hardy[15]), and was previously conjectured for a field theory in ref. 8.

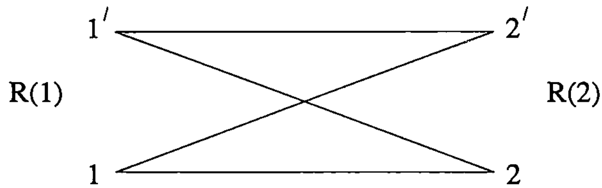

Fig. 3. The impossibility argument: 1′ is in the future of 1, and 2′ in the future of 2, the two pairs being spacelike separated. The lines indicate the four spacelike surfaces that can be drawn through these sites.

In general, then, the condition that (3) holds for all surfaces σ must be relaxed. In fact, for operational equivalence with standard quantum theory, only much weaker conditions are required—ones which can be satisfied consistently. To understand this, consider a general set of spacelike-separated variables $\alpha_{l_1},..., \alpha_{l_n}$. The joint distribution $p(\hat{\alpha}_{l_1},..., \hat{\alpha}_{l_n})$ only acquires *operational* meaning if the realized values $\hat{\alpha}_{l_r}$ can be compared, i.e., if there exist records of these values that can be brought together to the neighborhood of the same point (see ref. 8 for a similar point). For agreement with quantum theory, we therefore require as a first condition (I) on $p_\Psi(\hat{\alpha})$ that its *local* marginal distributions reproduce the quantum mechanical results. For our simple lattice theory, these local distributions are just the two-variable distributions associated with the ingoing pairs of links at each vertex. Of course this reasoning presupposes that the necessary records can in fact be made. Indeed, rather more generally, we require that under the appropriate circumstances (those associated with classical behavior in standard quantum theory) continuous, quasideterministic trajectories in the appropriate quantities should emerge. We are therefore led to a second condition (II) on $p_\Psi(\hat{\alpha})$: that it enforce sufficient continuity in time of the realizations $\hat{\alpha}_l$ to ensure that such trajectories appear.

We shall presently remove the imprecision in these heuristic remarks in the context of our specific model, but before doing so it is useful to illustrate them with two examples of realistic prescriptions in which there is a preferred frame and (3) is satisfied *only* on the constant-time slices σ_t. In the first, one simply picks a configuration on each σ_t independently, in accordance with the distributions $|\Psi(\hat{\alpha}|_{\sigma_t})|^2$. This prescription is Bell's (deliberately pathological) single-world version of the Everett theory.[16] Condition (I) is then satisfied, but not condition (II); the realizations on each slice occur with probabilities precisely according to the quantum prediction, but the independence of the slices means that they do not fit together to form sensible histories. The second example is Bell's stochastic generalization of the Bohm theory,[3] formulated for a fermionic field theory on a spatial lattice, with time kept continuous. Here, configurations on (infinitesimally close) time slices are connected by transition probabilities proportional to the transition amplitudes between the corresponding eigenstates. This transition rule (a generalization of Bohm's guiding condition) enforces the necessary continuity in time, so that (II) is satisfied as well. If one restricts attention to those quantities with quasiclassical behavior—which is the level at which the results of all measurements must be recorded—this is enough to ensure agreement (for these quantities) with the quantum results on all, and not just constant-time surfaces. One thus has a form of relativistic invariance, but it is phenomenological rather than fundamental. Our aim here is to go one step further and provide a scheme that is fully relativistic.

4. THE STOCHASTIC MODEL

Let the system be in the state Ψ. We describe a simple stochastic model which generates a joint distribution $p_\Psi(\hat{\alpha})$ with the same empirical content as the standard quantum theory, and for which no special set of surfaces is preferred. As will be explained later, the model may be regarded as the minimal realistic completion of the underlying quantum theory, and the extension to a rather general class of local lattice quantum theories is straightforward.

The initial conditions are a spacelike surface σ_0, the wavefunction $\Psi(\alpha|_{\sigma_0})$, and a configuration $\hat{\alpha}|_0$ on σ_0 chosen according to the quantum mechanical probability distribution (2), i.e., $|\Psi(\hat{\alpha}|_{\alpha_0})|^2$. (The dependence on σ_0 will be removed at the end of the construction by pushing it back into the infinite past.) The evolution of the wavefunction is taken to be the standard unitary one described before: thus in the simplest case—that of an elementary motion of the surface—a single R-matrix is applied as in (1). The specification of the dynamics is then completed by supplying a rule for the evolution of the configuration variables $\hat{\alpha}_l$. This is obtained very straightforwardly by imagining the initial surface to advance stochastically by successive random elementary motions, and requiring that given a particular motion, the joint distribution of the $\hat{\alpha}_l$ on the new surface is always given by the quantum mechanical result (2). Making the simplest independence assumption—that the realizations $\hat{\alpha}_l$ are otherwise random—this prescription is enough to uniquely define the model.[3]

[3] For a related idea, see the discussion of Bohmian theories in ref. 8 (and also ref. 9), where it is suggested that Bohm's guilding condition should be modified to act with respect to a dynamically determined foliation of spacetime. Note that here, where we do not have a guiding condition connecting the configurations on adjacent disjoint surfaces, but only the weaker constraint of consistency with (2), it will be essential that our surfaces are generated by elementary motions and are more "densely packed" than a foliation.

To be precise, label the lines of the lattice by L and R according to whether they represent left-moving or right-moving null rays, respectively. With a given surface σ_k, we may then associate a sequence $A|_k = (A_m)_{m=1}^{2N}$ of L's and R's labeling the successive links it cuts as one moves from left to right. Taking care to remember the periodicity in m, the prescription for the elementary motion to the next surface σ_{k+1} is: pick an RL pair from this sequence at random and move the surface up through the associated vertex, so that the pair is replaced by LR. As in Fig. 1, let this motion be from the links 1 and 2 to 1′ and 2′. To define the corresponding evolution of the configuration on σ_k, $\hat{\alpha}|_k$, we must specify the conditional probability $f_\Psi(\hat{\alpha}|_{\sigma_k \to \sigma_{k+1}})$ of realizing the values $(\hat{\alpha}_{1'}, \hat{\alpha}_{2'})$ on the newly cut links, given all realized values $\hat{\alpha}_l$ up to that point. If we make the simplest possible assumption—that there is no conditional dependence on the realizations to the past of σ_{k+1}—and apply the standard probability rule (2), then we obtain the unique prescription

$$f_\Psi(\hat{\alpha}|_{\sigma_k \to \sigma_{k+1}}) = \left.\frac{|\Psi(\alpha|_{\sigma_{k+1}})|^2}{\sum_{\alpha_{1'},\alpha_{2'}} |\Psi(\alpha|_{\sigma_{k+1}})|^2}\right|_{\alpha|_{\sigma_{k+1}} = \hat{\alpha}|_{\sigma_{k+1}}} \tag{5}$$

where, by unitarity, the denominator can also be written as $\sum_{\alpha_1,\alpha_2} |\Psi(\alpha|_{\sigma_k})|^2$.

Together, the rules for $A|_k$ and $\hat{\alpha}|_k$ define a discrete stochastic process $(A|_k, \hat{\alpha}|_k)$ $(k = 0, 1,...)$, and (summing over the A's) this generates a joint probability distribution $p_\Psi(\alpha|_{\sigma_\sigma^+})$ in the $\hat{\alpha}_l$ over the entire lattice (on and) to the future of the initial surface σ_0. Of course, the choice of a particular initial surface should be regarded as an artifact of the construction. The final step is thus to let σ_0 recede arbitrarily far into the past to obtain a distribution $p_\Psi(\hat{\alpha}) = \lim_{\sigma_0 \to -\infty} p_\Psi(\hat{\alpha}|_{\sigma_\sigma^+})$ independent of any choice of surfaces. The existence of this unique limiting distribution is shown in the Appendix.

The above is a dynamical description, in which $p_\Psi(\hat{\alpha})$ is regarded as being generated by successive realizations on an advancing spacelike front. It is also useful to think in a slightly different, but equivalent way, in terms of a probabilistic path integral. Consider a particular sequence of surfaces $\gamma = (\sigma_k)$ and let $p_\Psi^\gamma(\hat{\alpha}|_{\sigma_0^+})$ be the joint distribution in all the $\hat{\alpha}_l$'s to the future of σ_0 conditional on γ. By the above prescription we have

$$p_\Psi^\gamma(\hat{\alpha}|_{\sigma_0^+}) = |\Psi(\hat{\alpha}|_{\sigma_0})|^2 \prod_k f_\Psi(\hat{\alpha}|_{\sigma_k \to \sigma_{k+1}}) \tag{6}$$

Since in the stochastic process, σ_k evolves by random elementary motions, all possible γ contribute to the unconditional distribution $p_\Psi(\hat{\alpha}|_{\sigma_0^+})$ with equal weights. Thus,

$$p_\Psi(\hat{\alpha}|_{\sigma_0^+}) \sim \sum_\gamma p_\Psi^\gamma(\hat{\alpha}|_{\sigma_0^+}) \tag{7}$$

or, letting σ_0 recede arbitrarily into the past,

$$p_\Psi(\hat{\alpha}) \sim \sum_\gamma p_\Psi^\gamma(\hat{\alpha}) \tag{8}$$

where in each case, $\sim$ indicates the need for a normalization constant. If the $\hat{\alpha}_i$'s are thought of as sources, then (8) may be regarded as a sort of path integral, but involving probability weights rather than amplitudes.

The distribution $p_\Psi(\hat{\alpha})$ is most easily understood through the distributions $p_\Psi^\gamma(\hat{\alpha})$ conditional on γ. Each $p_\Psi^\gamma(\hat{\alpha})$ is straightforward to analyze since, *by construction*, its marginal distributions on all the $\sigma_k \in \gamma$ are just the quantum mechanical ones, i.e.,

$$\text{for each } \gamma: \qquad \sum_{\hat{\alpha}|_{\sigma_k^c}} p_\Psi^\gamma(\hat{\alpha}) = |\Psi(\hat{\alpha}|_{\sigma_k})|^2 \qquad \forall \sigma_k \in \gamma \tag{9}$$

If it can be shown that some prediction follows from $p_\Psi^\gamma(\hat{\alpha})$, independently of the choice of γ, then it is also true of the distribution $p_\Psi(\hat{\alpha})$.

5. SOME PROPERTIES OF THE MODEL

The constraint (9) satisfied by the $p_\Psi^\gamma(\hat{\alpha})$ is extremely powerful. It ensures that the model has the same predictive content as the standard quantum theory. In particular, it is straightforward to verify that the model satisfies the conditions (I) and (II) which arose in our earlier heuristic discussion.

Condition (I) follows from the fact that for every γ, each pair of ingoing links at a vertex (and, indeed, each outgoing pair, too) always lies on some $\sigma_k \in \gamma$. Thus the local marginal distributions of $p_\Psi(\hat{\alpha})$ associated with these pairs are just the quantum mechanical ones. It is worth remarking that by virtue of this, they automatically satisfy locality—that is to say each such distribution is independent of the R-matrices at spacelike-separated points. This may be seen explicitly by recalling that, as a result of local unitarity, such a distribution may be written as the appropriate marginal distribution of $|\Psi(\alpha|_\sigma)|^2$ for *any* σ cutting the relevant pair; pushing σ back in time as far as possible, all R-matrices spacelike separated from the pair will then lie in σ's future, thus making manifest their irrelevance to the pair's distribution.

To understand how the model satisfies the continuity condition (II), the crucial point to note is that the surface σ_k evolves by *local* (i.e., elementary) motions: given an appropriate conservation law, this automatically produces continuous trajectories. Thus, suppose that, as in Fig. 2b, the

R-matrices conserve occupation number, and that the state Ψ is an eigenstate of the total occupation number (sum of α_l's on an arbitrary surface). Then by (5), or equally (9), the *realized* occupation number will be conserved through an elementary motion and the stochastic process will generate $\hat{\alpha}$-configurations that are continuous on the lattice. Thinking of $\hat{\alpha}_l = 1$ on a link as the presence of a "particle," any given realization will thus be a set of continuous "particle paths" through the lattice. Moreover, the behavior of these paths will be straightforward to analyze, since for any γ, they will be cut by a "dense" sequence of surfaces $\sigma_k \in \gamma$, on which, by (9), the standard quantum prescription may be applied.

In the particular case of the lattice massive Thirring model, $\hat{\alpha}_l = 1$ corresponds to the presence of a "bare fermion." Going toward the continuum limit, and considering the nonrelativistic regime, the continuity property will carry over to the paths of the physical fermions as well.[4] If, for instance, a single particle is described by a localized wave packet in the standard fashion, then the preceding comments imply that in the stochastic model there will be a realized particle trajectory which follows the motion of this packet. Should the packet divide into several smaller packets, the trajectory will follow one of them with a probability given by the standard quantum mechanical probability of finding the particle in that particular branch. The motion of a many-particle system will be guided in the same way.

6. MEASUREMENT

To help to clarify these points and to make explicit the operational equivalence with the standard quantum theory, it is useful to consider a simple model of a measurement. Let us introduce then a second set of "apparatus" variables β_l on the links, again with the values zero and one. The R-matrices will now be eight-dimensional and in the state Ψ the stochastic model will generate a distribution $p_\Psi(\hat{\alpha}, \hat{\beta})$.

Suppose first that the β_l are completely independent of the α_l, i.e., that all the R-matrices of the complete system factorize as $R = R_\alpha \otimes R_\beta$, using an obvious notation. At every vertex let the only nonzero β-amplitudes be those depicted in Fig. 2b, but with the last two amplitudes which allow for a change of direction now also set to zero. Most simply, for instance, we may assume that at each vertex

$$R_{\beta_1 \beta_2}^{\beta_{1'} \beta_{2'}} = \delta_{\beta_1 \beta_{2'}} \delta_{\beta_2 \beta_{1'}} \tag{10}$$

[4] To be more precise, the (physical) particle position will only be well defined on scales greater than the Compton wavelength. The particle paths will be obtained by spacetime coarse-graining the $\hat{\alpha}_l$'s over this length scale and subtracting the corresponding coarse-grained density in the vacuum state.

Then, if the system is described by a joint eigenstate of the β-occupation number operators on some surface, the stochastic rule will generate a deterministic evolution in the $\hat{\beta}_l$ consisting of a set of "$\hat{\beta}$-particles" moving along null rays, independently of the realizations $\hat{\alpha}_l$.

Now suppose we wish to "measure" $\hat{\alpha}_l$ on a particular link. We modify the amplitudes for the joint system at the vertex v from which this link is outgoing, so that

$$R^{\alpha_{1'}\alpha_{2'}\beta_{1'}\beta_{2'}}_{\alpha_1\alpha_2 0\,0} = R^{\alpha_{1'}\alpha_{2'}}_{\alpha_1\alpha_2}\,\delta_{\alpha_{1'}\beta_{1'}}\delta_{\alpha_{2'}\beta_{2'}} \tag{11}$$

and choose the other matrix elements at v, $R^{\alpha_{1'}\alpha_{2'}\beta_{1'}\beta_{2'}}_{\alpha_1\alpha_2\beta_1\beta_2}$, to be consistent with unitarity. Let the system be in the state corresponding to all β-occupation numbers being zero on a surface σ prior to v, that is, again using an obvious notation, $|\Psi\rangle_\alpha \otimes |0\rangle_{\beta|_\sigma}$. Then all the β_l will realize the value zero except possibly on the two null rays beginning at v, and on these, the β-variables will realize the value one if and only if the corresponding variables $\alpha_{1'}$, $\alpha_{2'}$ do. To talk picturesquely, the realization of $\hat{\alpha}_l = 1$ on one of these links causes the emission of a $\hat{\beta}$-ray along the future null extenson of that link. This is our model of a measurement.

We now use this apparatus to determine the joint distribution of the $\hat{\alpha}$-variables at two spacelike-separated links l and l'. We shall show that the result obtained is necessarily equal to the standard quantum mechanical prediction; the extension to an arbitrary number of variables is then straightforward. We set up two measuring devices at the appropriate vertices, and suppose furthermore that the two links are inward-pointing, so that if $\hat{\beta}$-rays are produced, they will intersect at some point x in the future (see Fig. 4). (Note that this is precisely the situation we argued was necessary for the joint distribution to be given operational meaning; the

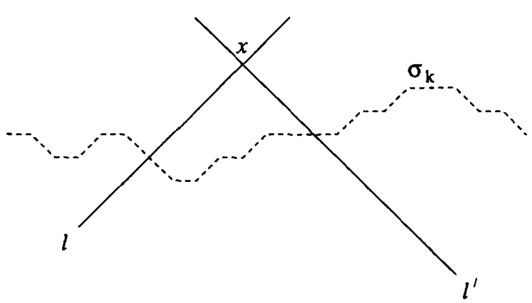

Fig. 4. Analysis of a measurement. The solid lines denote the $\hat{\beta}$-rays produced by the realization $\hat{\alpha}_l = 1$ and $\hat{\alpha}_{l'} = 1$. The dashed line is a surface σ_k picked out by a particular realization of the stochastic process.

$\hat{\beta}$-rays are functioning here as the relevant records.) By construction of the apparatus, the joint probability for the $\hat{\alpha}$-variables at the chosen links, $p(\hat{\alpha}_l, \hat{\alpha}_{l'})$, is equal to the joint probability for the production of $\hat{\beta}$-rays, and this in turn may be obtained from the marginal distribution for the $\hat{\beta}_l$ on any surface intersecting the causal past of the point x. But for any γ there will be at least one such surface belonging to γ, and so by (9) the agreement of the measured distribution with the quantum mechanical prediction then follows immediately. [Alternatively, simply apply (I) to the $\hat{\beta}_l$ on the pair of ingoing links at x.]

In fact, it should be clear that the same result holds rather more generally. It is not necessary that the records actually be brought together to the same point—only that, for every γ, there is a surface $\sigma_k \in \gamma$ cutting them, so that (9) may then be invoked. This circumstance will obtain provided that the records are sufficiently persistent. On the other hand, if it does not—as might happen, for instance, if a record is prematurely destroyed—then *a fortiori* there is some surface that separates the records, contact between them is excluded, and the joint distribution is deprived of operational significance. Harmony with the standard predictions is thus always maintained. Exactly the same considerations apply immediately to the measurement of the joint distribution $p(\hat{\alpha}_{l_1},..., \hat{\alpha}_{l_n})$ of an arbitrary set of spacelike-separated variables.

Of course the simplicity of the field theory and the demands of explicitness meant that our model of the measuring apparatus is rather crude. In practice, the appropriate amplitude structure leading to the continuous, quasideterministic trajectories of the classical regime will generally emerge from the quantum mechanical evolution law for systems with large numbers of degrees of freedom (decoherence) rather then being imposed by fiat at the microscopic level as above. The principle, however, remains precisely the same.

7. REMARKS

As we have already observed, the local (two-variable) marginal distributions satisfy locality. By contrast, a marginal distribution associated with a more extended region of spacetime will generally depend on the R-matrices at points spacelike separated from that region. This is a rather strong form of the nonlocality that we expected on general grounds from the outset. Nonetheless, because of the operational equivalence of the model with standard quantum theory, pathologies such as superluminal signaling are avoided, and the "peaceful coexistence"[17] of quantum theory and special relativity is left undisturbed. It is perhaps interesting to note that models of the second type (in which the unitary evolution is stochastically modified) are able to enjoy a better locality property in which the marginal distribution of events associated with a region is completely independent of the evolution law in spacelike-separated regions (though of course still nonlocally correlated with other events in those regions).[18,19] The stronger nonlocality we have found here is presumably the consequence of adhering to a strictly unitary evolution law for the wavefunction.

In spite of this nonlocality, it is worth emphasizing that the relativistic setting is crucial for the viability of the model. Were the underlying quantum theory nonrelativistic, then the only allowed "elementary motion" would be between adjacent constant-time slices, and the rule (5) would reduce simply to (2), with variables on different slices being independent. The model would thus collapse to Bell's version of Everett,[16] complete with its pathological absence of sensible histories. Indeed, one may think of the model as simply this Bell–Everett prescription, but with its two defects —absence of sensible histories and frame dependence—simultaneously cured by the use of the random sequence of surfaces that the locality of the evolution law of the quantum theory makes possible.

Another, rather suggestive way of thinking about the model is in terms of a stochastic accumulation of events on an advancing spacelike front—a picture somewhat like that proposed by Haag.[20] To make this more precise, define an "event" as the pair of realizations on the outgoing links at a particular vertex. The sum (8) is over all possible total orderings of events consistent with the partial ordering defined by the causal structure, so that an event may occur only after all the events in its causal past. Given such an ordering, the rule (5) simply describes the conditional probability of the realizations $\hat{\alpha}_l$ associated with an event, given all the events up to that point. Furthermore, (5) is the simplest assumption compatible with standard quantum mechanics, since it assumes a conditional dependence only on the other events of the current spacelike boundary, σ_{k+1}; any other rule would involve an additional dependence on the events to the past of this boundary, and so require an extension of the basic prescription (2). In this sense, the model is the minimal realistic completion of the underlying quantum theory.

It is tempting to speculate that by examining the distribution of events sufficiently far back in time, one could infer the state Ψ to arbitrary accuracy. One could then regard the quantum state and unitary evolution as purely auxiliary concepts, and think of the stochastic law governing the evolution as being one in which the probability of "the next event" depends simply on all the events which have ever preceded it (see also refs. 21 and 11 for related points).

8. GENERALIZATION

To conclude, we note that in constructing the stochastic model we have only made use of the causal structure and the local unitary evolution law of the underlying lattice field theory, together with a particular application of the standard probability interpretation of the wavefunction. There is therefore an immediate generalization to a broad class of such theories.

Consider a locally finite, partially ordered set of points $\mathscr{P}$ with the partial ordering $x < y$ describing the causal relation "y is in the future of x," and take for a lattice the corresponding Hasse diagram. (See, e.g., Stanley[22] for definitions, but note that we are using the word "lattice" in a nontechnical sense.) To each link l assign a number n_l of states, labeled by α_l ($=0,..., n_l - 1$), say, and suppose, as a condition on $\mathscr{P}$, that this can be done so that, at each vertex, the number of ingoing states equals the number of outgoing ones. By associating with the vertices unitary matrices connecting these states, one then obtains a local quantum theory on the lattice.

In this general framework, a spacelike surface σ is a cut through the links that intersects each everywhere future-directed path exactly once. At a vertex all of whose ingoing links are cut by σ, an elementary motion can be performed by moving the cut to the outgoing ones. One may thus associate a stochastic process with the quantum theory in exactly the same way as before—namely, by allowing a surface to evolve by random elementary motions, and taking the probabilistic law for the realizations on the newly cut links to be the obvious generalization of (5)—and again as before, this process may be used to generate a joint probability distribution $p_\Psi(\hat{\alpha})$ for the realizations $\hat{\alpha}_l$ over the lattice.

In principle it should be possible to construct a similar process for any local quantum theory whose evolution law can be regularized on a suitable causal lattice. In the case of a gauge theory, for instance, this will presumably involve group-valued variables on the links and a local evolution law associated with plaquettes rather than vertices. An important further question of course is whether one can obtain well-defined continuum limit. Needless to say, a formulation that employed continuum concepts from the outset, to be regularized in a convenient way at a second stage, would be highly desirable.

APPENDIX

To show the existence of the distribution $p_\Psi(\hat{\alpha})$ in the limit that σ_0 is pushed back arbitrarily into the past, it is useful to introduce the idea of the "time" t_σ associated with a surface σ. The flat surfaces σ_t (see Fig. 1) define a time coordinate t, whose units we can choose so that the time between adjacent surfaces is 1/2. For a general surface σ, we then define t_σ to be the average of the times of each of its links. Note that the sequence $(A_m)_{m=1}^{2N}$ associated with a surface σ defines its time up to an integer T $(=[t_\sigma])$; thus any surface is uniquely specified by the data $((A_m), T)$.

The random evolution law for σ_k corresponds to a homogeneous Markov process $(A|_k)$ $(k=0, 1,...)$: at each step an RL pair is chosen at random and replaced by LR. Moreover, the process is finite and irreducible. Consequently, there is a unique stationary distribution, and, as $t_{\sigma_0} \to -\infty$, the distribution over the space of sequences of surfaces between arbitrary finite times tends to a unique limiting distribution. Furthermore, *given* a particular sequence $\gamma = (\sigma_k)$, we have: (i) the sequence of realizations $(\hat{\alpha}|_k)$ is also a Markov process, since the conditional probability of $\hat{\alpha}|_{k+1}$ given all previous realizations depends only on $\hat{\alpha}|_k$, through (5); and (ii) the absolute probability of $\hat{\alpha}|_k$ is always $|\Psi(\hat{\alpha}|_{\sigma_k})|^2$. It is then immediate that, in any finite region of the lattice D, there is also a unique limiting distribution in the $\hat{\alpha}_l$. To generate it, simply take any time t for which all σ with $t_\sigma = t$ lie to the past of D, and use as initial conditions the distributions $|\Psi(\hat{\alpha}|_\sigma)|^2$ on each such σ, weighted according to the stationary distribution for the surfaces.

[For completeness note that the periodicity of the lattice means that any allowed $(A_m)_{m=1}^{2N}$ must have an equal number of R's and L's, and so the Markov process $(A|_k)$ is over a space of $(2N)!/N!N!$ states. It is also straightforward to show that each state has period $2N$, and appears in the stationary distribution with a weight given by the number of RL pairs it contains.]

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